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Two Very Accurate and Efficient Methods for Computing Eigenvalues and Eigenfunctions in Porous Convection Problems*

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We develop the compound matrix method and the Chebyshev tau method to be applicable to linear and nonlinear stability problems for convection in porous media, in a natural way. It is shown how to obtain highly accurate answers to problems which may be stiff, and spurious eigenvalues are avoided. A detailed analysis is provided for a porous convection problem of much current interest, namely convection with a horizontally varying temperature gradient. © 1996 Academic Press, Inc.

1. INTRODUCTION

Convective flow of a fluid in a porous medium is a subject driven by the immense variety of applications from which it arises. These applications are in biological, environmental, geophysical, and industrial contexts, to mention some. For example, convective motion of air in a layer of snow may be modelled by multiphase convection in a porous medium, Powers *et al.* [19]. The curious formation of stones into regular patterns in what is known as polygonal ground formation is believed to have its origins in thermally driven porous convection; see, e.g., Straughan [20, Chap. 7], and the references therein. The thawing of the permafrost layer below the sea bed off the coast of Alaska is a two phase convection in porous media process involving salt transport and thermal convection; see, e.g., Straughan [20, Chap. 7], and the references therein. Other examples of convective motion in porous media may be found in thermally driven cavities near the Earth's surface; or the spread of radioactive material which has leaked, the convection then being due to dissipation by a heat source. There are many other examples of convection in porous media which may be found in texts, e.g., Nield and Bejan [15], Straughan [20, 21].

One technique which has proved extremely valuable in assisting studies of convection in porous media is the theory of stability, whether by linear instability theory, nonlinear energy stability theory, or weakly nonlinear theory. Many recent studies in this field concentrate on applications; see, e.g., Nield [14], Nield *et al.* [16], and the many references in the books of Nield and Bejan [15] and Straughan [20, 21]. Stability theory is thus very important. Almost invariably stability calculations involve determining eigenvalues and eigenfunctions and few of the associated eigenvalue problems are solvable analytically. Hence accurate and efficient numerical eigenvalue/eigenfunction solvers are needed. In this paper we describe a new implementation of two existing techniques. These are the compound matrix method (Drazin and Reid [4], Ng and Reid [11–13]) and the Chebyshev tau method (Fox [5], Orszag [17]); we show how one may apply these techniques directly to the relevant equations of porous medium stability theory rather than convert to higher order equations or do other tricks. Thus, we show how the two techniques may be logically and naturally applied to many eigenvalue problems of linear and nonlinear stability theory in porous media convection. The methods we describe are very accurate, relatively easy to implement, and designed to handle variable coefficients and to avoid roundoff error and the appearance of spurious eigenvalues.

To motivate the methods we briefly consider the equations for convection in a porous medium. A specific example is included in Section 4, which investigates horizontal gradient convection, this example being selected to illustrate the complexity of equations we can handle. The standard equations for simple convective fluid motion in a porous elastic solid without other effects, may be written (cf. Straughan [20, p. 57])

$$p_{,i} = -\frac{\mu}{k} v_i - \delta_{i3}\rho_0 g(1 - \alpha[T - T_0]), \quad (1.1)$$

$$v_{i,i} = 0, \quad (1.2)$$

$$T_{,t} + v_i T_{,i} = \kappa \Delta T, \quad (1.3)$$

where $\mathbf{v}$ is seepage velocity, T is temperature, μ is dynamic

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viscosity, k is the permeability of the porous medium, α is the coefficient of thermal expansion of the fluid, g is gravity, ρ_0 (constant) is the density which is governed by the Boussinesq approximation, and κ is the thermal diffusivity. Standard indicial notation, together with the Einstein summation convention is employed. If these equations are considered in the layer $\{z \in (0, d)\} \times \mathbf{R}^2$ and the temperatures on $z = 0, d$ are prescribed and constant with values T_l, T_u, respectively, and the normal component of velocity vanishes on the planes $z = 0, d$, then the conduction solution is

$$\overline{T} = -\beta z + T_l, \quad \overline{v}_i = 0. \tag{1.4}$$

Under a suitable nondimensionalisation the perturbations to (1.4) may be shown to satisfy (see, e.g., Straughan [20, p. 57])

$$\begin{aligned} \pi_{,i} &= R\theta\delta_{i3} - u_i, \\ u_{i,i} &= 0, \\ \theta_{,t} + u_i\theta_{,i} &= Rw + \Delta\theta, \end{aligned} \tag{1.5}$$

where $(\mathbf{u}, \theta, \pi)$ is the perturbation to the steady solution $(\overline{\mathbf{v}}, \overline{T}, \overline{p})$, $w = u_3$, and where R^2 is the Rayleigh number. Equations (1.5) hold in the infinite three-dimensional layer contained in $0 < z < 1$, and the boundary conditions are

$$w = 0, \quad \theta = 0, \quad \text{on } z = 0, 1. \tag{1.6}$$

For Eqs. (1.5), (1.6) the theory of linear instability and nonlinear stability yield the same critical Rayleigh number which is a strong result. If one assumes the perturbations satisfy a plane tiling periodic (x, y) planform then in terms of a wave number a, system (1.5) gives rise to the eigenvalue problem

$$\begin{aligned} (D^2 - a^2)W + Ra^2\Theta &= 0, \\ (D^2 - a^2)\Theta + RW &= 0, \end{aligned} \tag{1.7}$$

together with the boundary conditions

$$W = 0, \quad \Theta = 0, \quad z = 0, 1, \tag{1.8}$$

where $D = d/dz$, and W, Θ denote the z-dependent parts of w, θ. Of course, (1.7), (1.8) may be solved exactly; however, they show why we are interested in studying an eigenvalue problem for a system of second-order differential equations. In general, stability studies in porous convection problems yield more complicated systems, possibly of higher order. Nevertheless, the basic ideas we study here are encompassed in (1.7). We stress that we do *not* convert (1.7) to a fourth-order system; it is better to work with the natural equations for the velocity and temperature fields *directly*. Interestingly, Gardner *et al.* [6] have advocated a Chebyshev numerical scheme which reduces a fourth-order equation to two second-order ones; for porous convection problems one is naturally faced with such a system. Our paper deals with a different issue to that of Gardner *et al.* [6], however, and we return to this in Section 4. Unlike Gardner *et al.* [6] we do not need systems of fourth-order equations; in porous convection problems we deal directly with (possibly many) systems of second-order equations. By dealing directly with only second-order equations we encounter only second-order differentiation operators and the matrices in the Chebyshev method grow no more than $O(N^3)$, where N is the number of polynomials. This is an important factor leading to accuracy because roundoff error is avoided. If a fourth-order differentiation operator is involved then the growth is $O(N^7)$, for sixth-order derivatives the growth is $O(N^{11})$, and so on; for Couette and Poiseuille like problems, which for high Reynolds number calculations 500 or more polynomials may be required, this is a serious problem. Restricting to second-order systems is a major asset in these problems even if multi-component diffusion or multi-layer flows are under investigation, as studied by Dongarra *et al.* [6] and Straughan and Walker [23].

To conclude the introduction we argue that as porous convection problems are becoming more complex it is useful to have two entirely different but nevertheless very accurate and efficient methods at ones disposal. When coefficients are complex and functions of the spatial variables and when the eigenvalues and eigenfunctions are complex, as they are, for example, in penetrative convection in an anisotropic porous medium with principal axis neither horizontal nor vertical Straughan and Walker [22]) then an independent check is extremely valuable.

2. THE COMPOUND MATRIX METHOD

The compound matrix method is designed to avoid round off error and works well if the system of differential equations is stiff. Its general history may be found in Drazin and Reid [4] and Ng and Reid [11]; these writers pay particular attention to a single fourth-order equation and the technique is extensively developed for this case with attention being given to the famous Orr–Sommerfeld problem, an equation well known for its difficulties. This method has been extended to an inhomogeneous fourth-order equation in Ng and Reid [12], and to sixth-order equations in Ng and Reid [13]. The method as advocated by Drazin, Ng, and Reid has been employed very successfully in several of the references quoted in this work to determine eigenvalues. One of the objectives of this work is to show how the ideas of Ng and Reid [11] may be extended directly to systems governing porous flow stability in a natural manner to find eigenfunctions. The eigenfunctions in turn yield streamlines and isotherms and so are very useful.

Consider the general linear system

$$\begin{aligned} w'' &= \alpha_1 w' + \alpha_2 w + \alpha_3 \theta' + \alpha_4 \theta, \\ \theta'' &= \beta_1 w' + \beta_2 w + \beta_3 \theta' + \beta_4 \theta, \end{aligned} \tag{2.1}$$

where a prime denotes differentiation with respect to x (in the porous convection case this is the variable z), α_1, ..., α_4, β_1, ..., β_4, are known coefficients which may depend on x, and may be complex, and $x \in (0, 1)$. One or more of the coefficients contains an eigenvalue, σ say. For definiteness, we limit the present discussion to the boundary conditions

$$w = \theta = 0 \quad \text{at } x = 0, 1, \tag{2.2}$$

although this is not necessary; other boundary conditions may be considered.

We could always eliminate one or other dependent variable in (2.1) and arrive at a single equation; for example, we here use w,

$$w^{\mathrm{IV}} = a_1 w''' + a_2 w'' + a_3 w' + a_4 w. \tag{2.3}$$

A similar equation may be derived for θ and then the theory of Ng and Reid [11] may be applied. The compound matrix method for this equation introduces the variables

$$\begin{aligned} u_1 &= w_1 w_2' - w_2 w_1', \\ u_2 &= w_1 w_2'' - w_2 w_1'', \\ u_3 &= w_1 w_2''' - w_2 w_1''', \\ u_4 &= w_1' w_2'' - w_2' w_1'', \\ u_5 &= w_1' w_2''' - w_2' w_1''', \\ u_6 &= w_1'' w_2''' - w_2'' w_1''', \end{aligned}$$

and works directly with these variables. One thing this does is that it avoids roundoff errors associated with interpolating on the zero of a determinant. The variables u_i satisfy the differential equation

$$\mathbf{u}' = B\mathbf{u},$$

where B is the 6×6 matrix given in terms of the a_i by Ng and Reid [11, p. 127].

With system (2.1) it is more natural to work with the 2×2 minors arising from w and θ, i.e.,

$$\begin{aligned} y_1 &= w_1 w_2' - w_2 w_1' \; (= u_1), \\ y_2 &= w_1 \theta_2 - w_2 \theta_1, \\ y_3 &= w_1 \theta_2' - w_2 \theta_1', \\ y_4 &= w_1' \theta_2 - w_2' \theta_1, \\ y_5 &= w_1' \theta_2' - w_2' \theta_1', \\ y_6 &= \theta_1 \theta_2' - \theta_2 \theta_1'. \end{aligned} \tag{2.4}$$

The y's satisfy the system

$$\mathbf{y}' = A\mathbf{y}, \tag{2.5}$$

where A (which is different from the B matrix of Ng and Reid [11]) is given by

$$A = \begin{pmatrix} \alpha_1 & \alpha_4 & \alpha_3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ \beta_1 & \beta_4 & \beta_3 & 0 & 1 & 0 \\ 0 & \alpha_2 & 0 & \alpha_1 & 1 & -\alpha_3 \\ -\beta_2 & 0 & \alpha_2 & \beta_4 & \alpha_1 + \beta_3 & \alpha_4 \\ 0 & -\beta_2 & 0 & -\beta_1 & 0 & \beta_3 \end{pmatrix}.$$

The eigenvalue(s) σ may be found by integrating (2.5) from 0 to 1 with the initial condition

$$y_5(0) = 1, \tag{2.6}$$

and we iterate on the final condition

$$y_2(1) = 0, \tag{2.7}$$

to derive σ to some preassigned degree of accuracy.

While the determination of σ is straightforward it is not so clear how to find the corresponding eigenfunction (w, θ). We could always convert to two fourth-order equations and use the theory of Ng and Reid [11] twice. However, it is preferable to avoid the need to construct the matrix B and the system for $\mathbf{u}$. When the coefficients in (2.1) are complicated this is a messy process. A self-contained process is desirable. It is worth pointing out that while (1.7) is very easy to handle, if we include effects like anisotropy and penetrative convection, then strong boundary layers may form and the system becomes stiff.

Let us observe that from (2.4) the y's satisfy the equation (cf. Ng and Reid [11])

$$y_1 y_6 - y_2 y_5 + y_3 y_4 = 0. \tag{2.8}$$

In fact, an analogous equation holds for the u's (Ng and Reid [11]). The theory of Ng and Reid [11] shows that the eigenfunction w to Eq. (2.3) may be found by backward integration of their equations (12)–(15). Ng and Reid [11] study the asymptotic behaviour of w near $x = 0$ and due to this there is a need to integrate in the 1 to 0 direction; however, they also show that their Eq. (12) is preferable due to the asymptotic behaviour.

For system (2.5) we may use Eqs. (12) and (14) of Ng and Reid [11] to yield a first-order system from which we can, in principle, determine the eigenfunctions directly. The system in our case is

$$\begin{aligned} y_2 w' - y_1\theta - y_4 w &= 0,\\ y_2\theta' - y_3\theta + y_6 w &= 0. \end{aligned} \tag{2.9}$$

Of course, we must verify (2.9) is equivalent to Eq. (12) of Ng and Reid [11] which in our notation is

$$u_1 w'' - u_2 w' + u_4 w = 0. \tag{2.10}$$

To establish the equivalence of the above two systems we differentiate $(2.9)_1$ to obtain

$$y_2 w'' + (y_2' - y_4)w' - y_4' w - y_1\theta' - y_1'\theta = 0.$$

Next substitute for θ' from $(2.9)_2$ and then eliminate θ from the resulting equation by using $(2.9)_1$ to obtain (provided $y_1, y_2 \neq 0$ in $(0, 1)$)

$$\begin{aligned} &y_2^2 y_1 w'' + (y_1 y_2 y_2' - y_4 y_1 y_2 - y_2^2 y_1' - y_1 y_2 y_3)w' \\ &\quad + (y_1^2 y_6 - y_4' y_1 y_2 + y_1' y_2 y_4 + y_1 y_3 y_4)w = 0. \end{aligned} \tag{2.11}$$

The coefficient of w'' is $y_2^2 u_1$. By using (2.5) we find the coefficients of w' and w are, respectively,

$$\begin{aligned} &-y_2^2(\alpha_1 y_1 + \alpha_4 y_2 + \alpha_3 y_3),\\ &y_2^2(-\alpha_2 y_1 + \alpha_4 y_4 + \alpha_3 y_5). \end{aligned} \tag{2.12}$$

But, $u_2 = w_1 w_2'' - w_2 w_1''$ and $u_4 = w_1' w_2'' - w_2' w_1''$ and, upon substitution from (2.1), we find

$$\begin{aligned} u_2 &= \alpha_1 y_1 + \alpha_3 y_3 + \alpha_4 y_2,\\ u_4 &= -\alpha_2 y_1 + \alpha_4 y_4 + \alpha_3 y_5. \end{aligned} \tag{2.13}$$

Thus, Eq. (2.11) is the same as Eq. (2.10).

It would be nice to simply integrate system (2.9). However, $y_2 = 0$ at $x = 0$ and $x = 1$. Fortunately, the method employed above yields a direct way to calculate the coefficients we need in (2.10). The point is that in integrating (2.5) we find the y's. We do not want to have to calculate B and then recalculate the u's. Hence, we may directly calculate w from Eq. (2.10) by integration backward from 1 to 0, with a knowledge of the y's; i.e., u_2, u_4 are determined from (2.13). Obviously, an analogous procedure exists for finding $\theta(x)$.

3. A CHEBYSHEV TAU METHOD FOR SYSTEM (2.1)

Let us begin by rewriting (2.1) and by defining the operators L_1 and L_2 by

$$\begin{aligned} L_1 u &= u'' - \alpha_0 u - \alpha_1 v - \alpha_2 u' - \alpha_3 v',\\ L_2 v &= v'' - \beta_0 u - \beta_1 v - \beta_2 u' - \beta_3 v'. \end{aligned} \tag{3.1}$$

The coefficients α_i, β_i may depend on x and be complex, and in addition the eigenvalue σ appears in one or more coefficients. We study the system

$$\begin{aligned} L_1 u &= 0,\\ L_2 v &= 0, \end{aligned} \tag{3.2}$$

on $(-1, 1)$, together with the boundary conditions

$$u = v = 0 \quad \text{at } x = \pm 1. \tag{3.3}$$

In traditional implementations of the Chebyshev tau method spurious eigenvalues have appeared and recent articles have been concerned with the removal of these (see Gardner *et al.* [6], Lindsay and Ogden [9], Zebib [24–26]). Gardner *et al.* [6] write that the spurious eigenvalues are due to singularities in the matrix associated with the eigenvalue which arises in the discrete representation of a differential equation. They concentrate on a fourth-order equation or a system of fourth-order equations. Their idea is to reduce a fourth-order equation to two second-order ones. They remove boundary condition rows and effectively reduce everything to a single equation for the discrete variable, Gardner *et al.* [6, Eq. (3.9b)]; if one goes carefully through their work it is not clear to the present writers that this procedure is not effectively equivalent to the original one of Orszag [17], with columns removed due to boundary conditions, because of the matrix B_1Q in (3.9b). This matrix has, in fact, $O(N^6)$ growth and so care must be taken with round off error. Nevertheless, the paper of Gardner *et al.* is very interesting and inspired the work of the present section. Use of Chebyshev polynomials in hydrodynamic stability problems has been advocated for several years, and the very fundamental paper of Orszag [17] has been a cornerstone in the field.

We now describe the procedure for finding eigenvalues and eigenfunctions to (3.2), (3.3), although we stress other boundary conditions may be handled, and higher order systems may be dealt with by the same technique. Thus, the method is applicable to many studies of stability in both porous media and hydrodynamics.

The underlying idea is to write u, v as a finite series of Chebyshev polynomials

$$u = \sum_{k=0}^{N+2} a_k T_k(x), \qquad v = \sum_{k=0}^{N+2} b_k T_k(x), \tag{3.4}$$

although the logic is that (3.4) are truncations of an infinite series. Due to the truncation, instead of solving (3.2) one solves

$$L_1 u = \tau_1 T_{n+1} + \tau_2 T_{N+2}, \qquad L_2 v = \hat{\tau}_1 T_{N+1} + \hat{\tau}_2 T_{N+2}, \tag{3.5}$$

where τ_1, τ_2, $\hat{\tau}_1$, $\hat{\tau}_2$ are parameters which may be used to measure the error associated with truncation in (3.4) (cf. the Lanczos technique and the error analysis of Fox [5]).

To obtain a resolvable problem the inner product with T_i is taken in (3.5) in the weighted $L^2(-1, 1)$ space with inner product

$$(f, g) = \int_{-1}^{1} \frac{fg}{\sqrt{1-x^2}}\, dx,$$

and associated norm $\|\cdot\|$. Since the Chebyshev polynomials are orthogonal in this space, from (3.5) we obtain $2(N + 1)$ equations

$$(L_1 u, T_i) = 0, \quad i = 0, 1, ..., N, \qquad (L_2 v, T_i) = 0, \quad i = 0, 1, ..., N. \tag{3.6}$$

The further conditions from (3.5),

$$(L_1 u, T_{N+j}) = \tau_j \|T_{N+j}\|^2, \quad j = 1, 2,$$

$$(L_2 v, T_{N+j}) = \hat{\tau}_j \|T_{N+j}\|^2, \quad j = 1, 2,$$

are useful error indicators and may indicate when spurious eigenvalues are present (cf. Gardner *et al.* [6]). Instead, four further conditions are found from the boundary conditions which since $T_n(\pm 1) = (\pm 1)^n$ (Orszag [17]) give

$$\sum_{n=0}^{N+2} (-1)^n a_n = 0, \quad \sum_{n=0}^{N+2} a_n = 0, \qquad \sum_{n=0}^{N+2} (-1)^n b_n = 0, \quad \sum_{n=0}^{N+2} b_n = 0. \tag{3.7}$$

Thus, (3.6) and (3.7) yield a system of $2(N + 3)$ equations for the $2(N + 3)$ unknowns a_i, b_i, $i = 0, ..., N + 2$. In this way, as Orszag [17] succinctly explains, the high frequency behaviour of the solution is determined not by the dynamical equations but rather by the boundary conditions. For ease of exposition we suppose α_i, β_i are constant, although in Section 4 we deal with situations where this is not the case; the general procedure necessary when α_i, β_i are functions of x may be developed with the aid of Orszag's [17 p. 702] relations.

Since the derivative of a Chebyshev polynomial is a linear combination of lower order Chebyshev polynomials,

$$T'_n = \begin{cases} 2n(T_{n-1} + \cdots + T_1), & n \text{ even}, \\ 2n(T_{n-1} + \cdots + T_2) + nT_0, & n \text{ odd}, \end{cases} \tag{3.8}$$

an expression which may be calculated from Eq. (A2) of Orszag [17], then remembering (3.6) are truncated versions of infinite expressions, (3.6) may be written (cf. Gardner *et al.* [6, p. 141]; Orszag [17, p. 693],

$$a_i^{(2)} - \alpha_0 a_i - \alpha_1 b_i - \alpha_2 a_i^{(1)} - \alpha_3 b_i^{(1)} = 0, \quad i = 0, ..., N, \qquad b_i^{(2)} - \beta_0 a_i - \beta_1 b_i - \beta_2 a_i^{(1)} - \beta_3 b_i^{(1)} = 0, \quad i = 0, ..., N, \tag{3.9}$$

where the new coefficients are given by

$$a_i^{(1)} = \frac{2}{c_i} \sum_{\substack{p=i+1 \\ p+i \text{ odd}}}^{p=N+2} p a_p, \tag{3.10}$$

$$a_i^{(2)} = \frac{1}{c_i} \sum_{\substack{p=i+2 \\ p+i \text{ even}}}^{p=N+2} p(p^2 - i^2) a_p, \tag{3.11}$$

with an analogous representation for $b_i^{(1)}, b_i^{(2)}$, and with $c_0 = 2$, $c_i = 1$, $i = 1, 2, ...$. If (3.9) are coupled together with (3.7) we may obtain a matrix equation

$$A\mathbf{x} = \sigma B\mathbf{x},$$

with $\mathbf{x} = (a_0, ..., a_{N+2}, b_0, ..., b_{N+2})^{\mathrm{T}}$. However, the B matrix is inevitably singular due to the way the boundary condition rows are added to A. This point is easily overcome; indeed the device is that used by Haidvogel and Zang [7] (fourth line after Eq. (7)) in another context. To see this observe that from (3.7), we easily eliminate $a_{N+1}, a_{N+2}, b_{N+1}, b_{N+2}$. For, suppose for definiteness N is odd, then

$$\begin{aligned} a_{N+1} &= -(a_0 + a_2 + \cdots + a_{N-1}) \\ a_{N+2} &= -(a_1 + a_3 + \cdots + a_N) \end{aligned} \tag{3.12}$$

with analogous expressions involving the b's, and thus the $N + 1$ and $N + 2$ rows of D, D^2 may be removed and the $N + 1$, $N + 2$ columns eliminated using (3.12). This yields $(N + 1) \times (N + 1)$ matrices D, D^2, and the matrix problem which results does not suffer from B being singular due to zero boundary condition rows.

The upshot is (2.2) is replaced by a system

$$A\mathbf{x} = \sigma B\mathbf{x} \tag{3.13}$$

where $\mathbf{x} = (a_0, ..., a_N, b_0, ..., b_N)$. Explicit details of A, B are given in Section 4 when the examples illustrate more clearly the method. The eigenvalues of (3.13) are found most efficiently using the QZ algorithm of Moler and Stewart [10] and this is implemented in many standard libraries, e.g., the routines ZGGHRD, ZHGEQV, and ZTGEVC of the LAPACK Fortran subroutine library, Anderson *et al.* [1], or the routines F02BJF, F02GJF of the NAG library. Care must be taken with this implementation if B is singular due to the nature of the differential system (3.2). Also, the eigenfunctions are efficiently computed via the QZ algorithm, with further details being provided explicitly in Section 4.

It is pertinent at this point to mention the work of Lindsay and Ogden [9] who develop a heuristic method for solving an arbitrary system of differential equations, generalizing the ideas of Gardner *et al.* [6]. Applied to (3.2) the idea of [9] is to rewrite the two equations as

$$\begin{aligned} u' &= w \\ w' &= \alpha_0 u + \alpha_1 v + \alpha_2 w + \alpha_3 p \equiv \Gamma_1 \\ v' &= p \\ p' &= \beta_0 u + \beta_1 v + \beta_2 w + \beta_3 p \equiv \Gamma_2. \end{aligned}$$

Although a tau method is not explicitly discussed in [9] one could solve

$$\begin{aligned} u' - w &= \tau_1 T_{N+1} \\ w' - \Gamma_1 &= \tau_2 T_{N+1} \\ v' - p &= \tau_3 T_{N+1} \\ p' - \Gamma_2 &= \tau_4 T_{N+1} \end{aligned}$$

and then obtain error indicators as

$$\begin{aligned} (u' - w, T_{N+1}) &= \tau_1 \|T_{N+1}\|^2 \\ (w' - \Gamma_1, T_{N+1}) &= \tau_2 \|T_{N+1}\|^2 \\ (v' - p, T_{N+1}) &= \tau_3 \|T_{N+1}\|^2 \\ (p' - \Gamma_2, T_{N+1}) &= \tau_4 \|T_{N+1}\|^2. \end{aligned}$$

While the technique advocated in [9] is not without interest we do not believe *the method by itself* leads to removal of spurious eigenvalues. For example, if we use it to solve the problem

$$y'' + \lambda y = 0, \quad x \in (-1, 1),$$
$$y(-1) = y(1) = 0,$$

then the recipe of [9] with

$$y = \sum_{n=0}^{N+1} y_n T_n(x), \quad v = \sum_{n=0}^{N+1} v_n T_n(x),$$

and

$$y' = v, \quad v' = -\lambda y,$$

requires the solution of

$$\begin{pmatrix} D & -I \\ BC1 & 0 \cdots 0 \\ & \\ 0 & D \\ BC2 & 0 \cdots 0 \end{pmatrix} \begin{pmatrix} \mathbf{y} \\ \mathbf{v} \end{pmatrix} = \lambda \begin{pmatrix} 0 & 0 \\ 0 \cdots 0 & 0 \cdots 0 \\ & \\ -I & 0 \\ 0 \cdots 0 & 0 \cdots 0 \end{pmatrix} \begin{pmatrix} \mathbf{y} \\ \mathbf{v} \end{pmatrix}, \tag{3.14}$$

where $BC1$, $BC2$ refer to the boundary condition rows

$$\sum_{n=0}^{N+1} (-1)^n y_n = 0, \quad \sum_{n=0}^{N+1} y_n = 0.$$

Using $N = 18$ we found of the 40 β values produced using the QZ algorithm, 21 were with $\beta = 0$. However, from the remaining list of 19 the smallest eigenvalue given by the method of [9] we computed as

$$\lambda = -0.137406 \times 10^{18}. \tag{3.15}$$

The next in the list is the correct one. Upon closer inspection, we found the QZ algorithm returns

$$\alpha_r = -0.261515 \times 10^3$$

and

$$\beta = 0.190322 \times 10^{-14}$$

(to six d.p.) for the value (3.15), where α_r is the real part of the variable α produced. This behaviour is repeated for other values of N. For example, for $N = 8$ we find 11 β's are zero and one set of α_r, β, λ such that

$$\alpha_r = 0.581989 \times 10^2, \quad \beta = 0.114396 \times 10^{-14},$$

$$\lambda = 0.508751 \times 10^{17}.$$

With $N = 17$, 19, and 28 we find, respectively, 20, 22, 31 of the β's are zero and

$$\alpha_r = -0.223482 \times 10^3, \quad 0.277560 \times 10^3, \quad 0.610930 \times 10^3,$$

$$\beta = 0.125559 \times 10^{-14}, \quad 0.595272 \times 10^{-15}, \quad 0.857495 \times 10^{-14},$$

$$\lambda = -0.177989 \times 10^{18}, \quad 0.466274 \times 10^{18}, \quad 0.712460 \times 10^{17}.$$

No such spurious behaviour was found when the D^2 method, i.e., that which is equivalent to the one leading to (3.13), was employed. Clearly care must be exercised with the method of [9]; this technique is further investigated (among others) in [2].

With the technique of [9] the boundary condition rows do not occur in a way which allows the removal of the y_{N+1}, v_{N+1} terms (unless the boundary conditions are very special). Moreover, the B matrix will always be singular. The numerical routine behind [9], i.e., the QZ algorithm, is employed to filter out those β_i which are zero. The QZ algorithm does not yield λ directly, rather a set α_i, β_i and if $\beta_i \neq 0$, $\lambda = \alpha_i/\beta_i$ gives the correct eigenvalues. The QZ algorithm may be used in this way if one is careful, although both the α_i and β_i should not both be close to zero together, and as our example shows, consideration of the α_i and β_i is essential.

We now illustrate the use of the compound matrix and Chebyshev tau methods in an important new physical problem of convection in a porous medium. This highlights the ability of these techniques to deal with coefficients dependent on the vertical spatial variable and complex coefficients, although more general boundary conditions and systems of order higher than 4 which occur naturally in practical porous convection problems can also be accommodated. The critical values presented in Tables I and II are obtained using the compound matrix method, together with secant and quasi-Newton techniques in the optimization. A Burlisch and Stoer extrapolation solver and a Runge–Kutta–Fehlberg technique were employed to numerically integrate the ordinary differential equations. However, all values, including those for the eigenfunctions, have been checked using the Chebyshev technique and the agreement is good. The degree of accuracy we obtain is largely controlled by the optimization routines. We can obtain five or six decimal places of accuracy in a reasonable amount of time and, by demanding smaller tolerances, greater accuracy may be achieved, although at the expense of computing time. However, the accuracy of the two methods is compared in detail in Section 4; it is seen that excellent agreement is established.

4. CONVECTION IN A POROUS MEDIUM WITH INCLINED TEMPERATURE GRADIENT

Nield [14] studies convection in a layer of porous material when there is a temperature gradient in the vertical direction but that gradient varies as one traverses the layer in one of the horizontal directions, the x-direction say. From our point of view this is an interesting example since the equations involve complex coefficients which depend on the z variable. Numerically we find this is a problem which for certain parameter ranges is very sensitive to small changes in parameters; hence it is a good test of a method's accuracy.

The work of Nield [14] is extended in Nield *et al.* [16] who investigate the analogous problem when temperature and salt fields are present. The equations may be conveniently found in Nield and Bejan [15, p. 257], but we take them from Nield [14] who employs a different nondimensionalisation. The layer of porous medium is subject to temperature fields on the boundaries $z = \pm H/2$ with

$$T = T_0 \mp \tfrac{1}{2}\,\Delta T - \beta_T x,$$

where ΔT is the temperature drop over the layer in the vertical direction and β_T is a constant. In terms of vertical and horizontal Rayleigh numbers R_V, R_H, which are physical parameters which are prescribed, the temperature boundary conditions in nondimensional form are

$$T = \mp\tfrac{1}{2}R_V - R_H x, \quad z = \pm\tfrac{1}{2}.$$

These boundary conditions lead to a steady solution in which the horizontal velocity is not zero; the steady solution has form

$$\overline{U} = R_H z,$$
$$\overline{T} = -R_V z + \tfrac{1}{24}R_H^2(z - 4z^3) - R_H x, \tag{4.1}$$

with $z \in (-\frac{1}{2}, \frac{1}{2})$.

The non-dimensionalised perturbation equations from this solution, written in normal mode form with k, m being the x and y wavenumbers, and $a^2 = k^2 + m^2$, are

$$\begin{aligned} (D^2 - a^2)W + a^2\Theta &= 0, \\ [D^2 - a^2 - i\sigma - ik\overline{U}(z)]\Theta + \frac{ik}{a^2}R_H DW - (D\overline{T})W &= 0, \end{aligned} \tag{4.2}$$

$z \in (-\frac{1}{2}, \frac{1}{2})$, and these equations are to be solved subject to

$$W = \Theta = 0, \quad z = \pm\tfrac{1}{2}. \tag{4.3}$$

The basic solution (4.1) is referred to by Nield [14] as Hadley flow.

Nield [14] argues that $\sigma = 0$ is sufficient, whereas we fix R_V and R_H and solve for σ. We minimize in k and m using a quasi-Newton routine to find the critical Rayleigh number. For fixed R_V we find that value of R_H such that $\sigma_r = 0$ ($\sigma = \sigma_r + i\sigma_i$). We only treat the case where R_V is in the decreasing phase of Nield's [14] analysis as this is where the eigenvalues/eigenfunctions behave most badly. We do confirm Nield's findings that $\sigma_i = 0$ at criticality.

The compound matrix equations arising from (4.2) are (we give the 12 equations which arise from the six complex equations)

$$\begin{aligned}
y_1' &= -a^2 y_2, \\
y_2' &= y_3 + y_4, \\
y_3' &= y_5 + (a^2 + \sigma_r)y_2 - (\sigma_i + kU)y_8 + \frac{k}{a^2}R_H y_7, \\
y_4' &= y_5 + a^2 y_2, \\
y_5' &= (a^2 + \sigma_r)y_4 - (\sigma_i + kU)y_{10} - \gamma y_1 + a^2(y_3 - y_6), \\
y_6' &= -\frac{k}{a^2}R_H y_{10} - \gamma y_2, \\
y_7' &= -a^2 y_8, \\
y_8' &= y_9 + y_{10}, \\
y_9' &= y_{11} + (a^2 + \sigma_r)y_8 + (\sigma_i + kU)y_2 - \frac{k}{a^2}R_H y_1, \\
y_{10}' &= y_{11} + a^2 y_8, \\
y_{11}' &= (a^2 + \sigma_r)y_{10} + (\sigma_i + kU)y_4 - \gamma y_7 + a^2(y_9 - y_{12}), \\
y_{12}' &= \frac{k}{a^2}R_H y_4 - \gamma y_8.
\end{aligned} \tag{4.4}$$

Here γ is the nondimensional temperature gradient,

$$\gamma = -R_V - \frac{R_H^2}{12}(1 - 6z + 6z^2),$$

where we have set (4.4) in the domain $z \in (0, 1)$, so

$$U = R_H(z - \tfrac{1}{2}).$$

System (4.4) is integrated subject to the initial condition

$$y_5(0) = 1$$

and final conditions

$$y_2(1) = y_8(1) = 0.$$

The Chebyshev scheme of Section 3 applied to (4.2); (4.3) reduces to solving

$$A\mathbf{x} = \sigma B\mathbf{x} \tag{4.5}$$

where now $\mathbf{x} = (W_0, ..., W_N, \Theta_0, ..., \Theta_N)$, and the matrices A and B are given by

$$A = \begin{pmatrix} D^2 - a^2 I & a^2 I \\ \frac{ik}{a^2}R_H D + \left(R_V - \frac{R_H^2}{24}\right)I + \frac{R_H^2}{8}P & D^2 - a^2 I - \frac{1}{2}ikR_H M \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.$$

In the expression for A, P is the matrix which arises from the Chebyshev representation of z^2. In the code we arrange for (4.5) to involve $N \times N$ matrices, so

$$\begin{aligned}
P_{11} = P_{ii} = \tfrac{1}{2}, \quad i = 3, ..., N; \quad P_{22} = \tfrac{3}{4}; \\
P_{i,i+2} = \tfrac{1}{4}, \quad i = 1, ..., N-2; \\
P_{31} = \tfrac{1}{2}; \quad P_{i+2,i} = \tfrac{1}{4} \quad i = 2, ..., N-2.
\end{aligned}$$

D^2 is the second differentiation matrix incorporating boundary conditions of type (4.3). It is worth observing that B above is singular and we find one half of the β_i given by the QZ algorithm are zero (as we should).

We find results in broad qualitative agreement with those of Nield [14], although the exact quantitative values we find are different. This is likely to be due to the fact that Nield employs a Galerkin argument. However, the differences are not major and we do agree with the streamline predictions at criticality. For the parameter range we have chosen, the eigenvalues and eigenfunctions are very sensitive to changes in the parameters. Nield [14] notes there is instability even when $R_V = 0$, i.e., when there is no vertical temperature difference, so the convection is driven

TABLE I

Critical Rayleigh and Wave Numbers

R_H	R_V	$a = m$
114.20	100	9.56398
120.05	75	9.99614
125.11	50	10.3557
133.73	0	10.8870
135.28	-10	11.0463

by the horizontal temperature distribution only. We see below that convection is possible even when $R_V < 0$, i.e., when the fluid is heated from above; this is not so surprising when one recalls convection in a slot can be driven by differentially heated sidewalls. Also, we find the onset of instability is with $k = 0$, agreeing with Nield's [14] analysis. Some values for the onset of convection are given in Table I.

In Figs. 1 and 2 the eigenfunctions $W(z)$ and $\Theta(z)$ corresponding to those values in Table I are presented. We could produce a sketch of the streamfunctions as is done by Nield [14], but this is easily visualised with the aid of Equation (21) of [14]. From Figs. 1 and 2 it is evident that the eigenfunction $W_1 = W(z)$ is an odd function across the layer for $R_V = 100$, 75, 50, while it is even for $R_V = 0$, -10. It ought to be observed that the curve for $R_V = 75$ is negative when $z \in (0, 0.5)$, but since the problem is linear we may take it to have the opposite sign. The same qualitative behaviour is true of the temperature eigenfunction $\Theta(z)$. The function W is actually negative in the region $z \in [0.45, 0.55]$ when $R_V = 0$ and when $R_V = -10$.

Nield [14] remarks that for $R_H = 110$ he sees two nearby eigenvalues and the eigenfunction for W is different and interesting in each case. The Chebyshev method yields as many eigenvalues as one wishes and so we present below the first two for two representative values for a value of R_V close to that discussed by Nield.

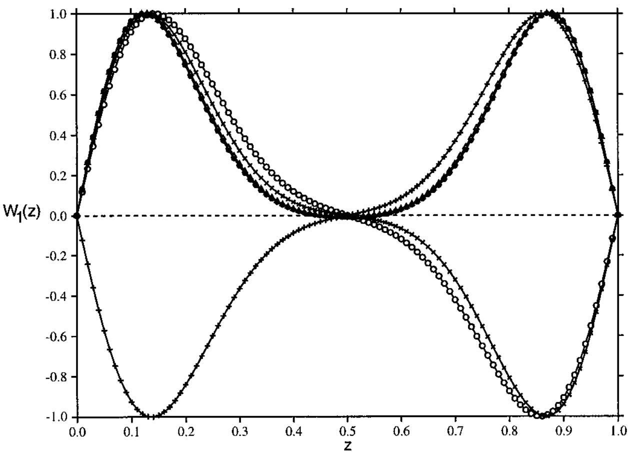

FIG. 1. Plots of $W_1(z)$: open circle (○), $R_V = 100$; plus (+), $R_V = 75$; cross (×), $R_V = 50$; triangle (△) $R_V = 0$; filled circle (●), $R_V = -10$.

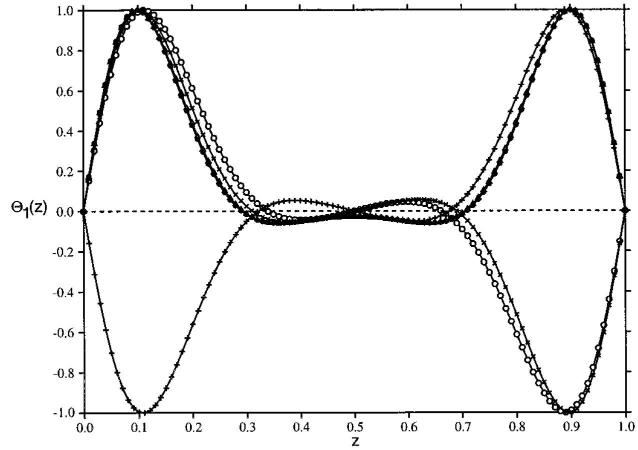

FIG. 2. Plots of $\Theta_1(z)$: open circle (○), $R_V = 100$; plus (+), $R_V = 75$; cross (×), $R_V = 50$; triangle (△), $R_V = 0$; filled circle (●), $R_V = -10$.

In Figs. 3 and 4 the eigenfunctions (W_1, Θ_1), (W_2, Θ_2) corresponding to σ^1 and σ^2 are presented. Nield does not give Θ graphs. We stress that with the Chebyshev method it is easy to generate as many eigenvalues and eigenfunctions as we wish. For both Figs. 3 and 4 the functions W_1, Θ_1 are odd functions of z (about $z = \frac{1}{2}$) while W_2, Θ_2 are even. The function W_2 in Fig. 3 is actually negative when $z \in [0.49, 0.51]$, with

$$W_2(0.5) = -0.17532 \times 10^{-2};$$
$$W_2(0.49) = W_2(0.51) = -0.12513 \times 10^{-2},$$

whereas in Fig. 4 W_2 is always positive, the minimum in the interior of the layer being

$$W_2(0.5) = 0.23081 \times 10^{-1}.$$

Numerical Comparison of the Compound Matrix and Chebyshev Tau Methods

We present a numerical comparison of the two methods for the system (4.2), (4.3). The results of Table I involve

TABLE II

Values of σ_i for the First Two Eigenvalues

R_V	σ_i^1	σ_i^2
50	0	-0.10729
100	0	-0.50695

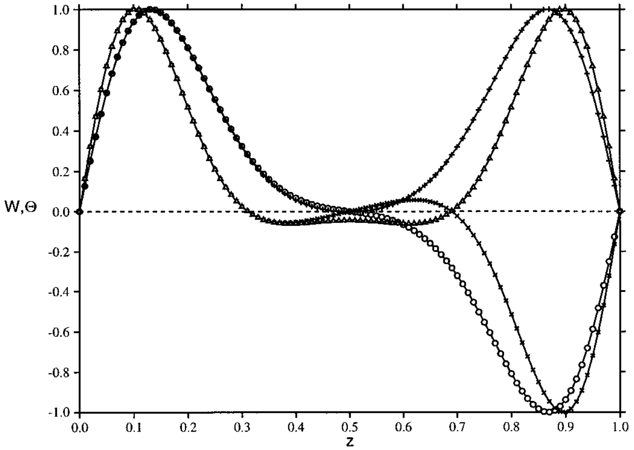

FIG. 3. Plots of first two eigenfunctions, $R_V = 50$: open circle (○), W_1; plus (+), W_2; cross (×), Θ_1; triangle (△), Θ_2.

optimization in the wavenumbers k and m and the accuracy of the results given there is effectively controlled by the accuracy we demand of the optimization routine. Thus, to see directly how the compound matrix and Chebyshev tau methods compare we present results for (4.2) and (4.3) with k, m, R_H, R_V fixed, and we then determine σ.

Table III represents values obtained by the compound matrix method. The tolerances odetol and sectol represent the accuracy required of the ODE solver and the secant method, respectively. It is seen that the ODE tolerance must be kept small.

Tables IV to VI were obtained via the Chebyshev tau method. For the values chosen the eigenfunction corresponding to $\sigma^{(1)}$ is real and odd and the τ coefficients are such that $|\tau_1| = |\hat{\tau}_1| = 0$. The eigenvector which solves (4.5) is normalised such that the sum of squares of the moduli

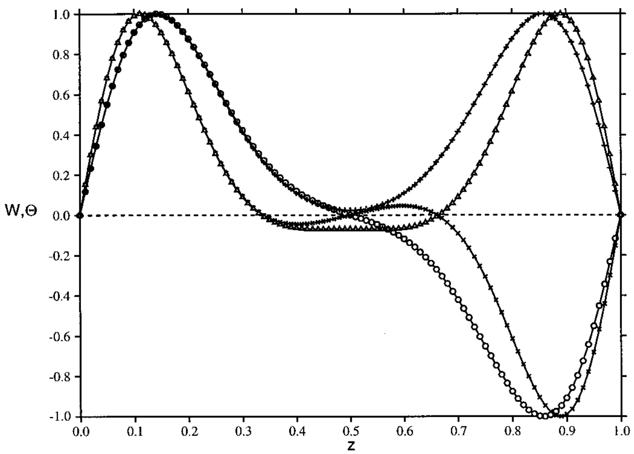

FIG. 4. Plots of first two eigenfunctions, $R_V = 100$: open circle (○), W_1; plus (+), W_2; cross (×), Θ_1; triangle (△), Θ_2.

TABLE III

The Real Part of the Leading Eigenvalue $\sigma^{(1)}$ against the Tolerances Odetol and Sectol: $R_H = 114.2$, $R_V = 100$, $k = 0$, $m = 10$

$\sigma_r^{(1)}$	Odetol	Sectol
−0.2934327661	10^{-14}	10^{-8}
−0.2934327663	10^{-12}	10^{-8}
−0.2934327680	10^{-10}	10^{-8}
−0.2934332109	10^{-8}	10^{-8}
−0.2934327661	10^{-14}	10^{-6}
−0.2934327663	10^{-12}	10^{-6}
−0.2934327680	10^{-10}	10^{-6}
−0.2934332109	10^{-8}	10^{-6}
−0.2934327661	10^{-14}	10^{-4}
−0.2934327662	10^{-12}	10^{-4}
−0.2934327680	10^{-10}	10^{-4}
−0.2934332109	10^{-8}	10^{-4}
−0.2934327658	10^{-14}	10^{-2}
−0.2934327659	10^{-12}	10^{-2}
−0.2934327676	10^{-10}	10^{-2}
−0.2934332106	10^{-8}	10^{-2}

TABLE IV

The Real Part of the Leading Eigenvalue $\sigma^{(1)}$ against the Number of Polynomials N

$\sigma_r^{(1)}$	W_r^N	Θ_r^N	N
−0.2911367321	-0.266737×10^{-2}	-0.338933×10^{-2}	12
−0.2934457207	0.615707×10^{-4}	0.234320×10^{-3}	16
−0.2934327257	-0.556424×10^{-6}	-0.422922×10^{-5}	20
−0.2934327661	0.525455×10^{-9}	0.308362×10^{-7}	24
−0.2934327661	0.129031×10^{-10}	$-0.481852 \times 10^{-10}$	28
−0.2934327661	$-0.700439 \times 10^{-13}$	$-0.258404 \times 10^{-12}$	32
−0.2934327661	0.295135×10^{-15}	0.140087×10^{-14}	36

Note. Here W_r^N is the Nth coefficient of the eigenvector representing $W^{(1)}$ and Θ_r^N has a similar meaning for $\Theta^{(1)}$: $R_H = 114.2$, $R_V = 100$, $k = 0$, $m = 10$.

TABLE V

The Real Part of the Leading Eigenvalue $\sigma^{(1)}$ and the Coefficients $|\tau_2|$ and $|\hat{\tau}_2|$, Defined after (3.6)

$\sigma_r^{(1)}$	$\lvert\tau_2\rvert$	$\lvert\hat{\tau}_2\rvert$	N
−0.2911367321	0.177958	0.896724	12
−0.2934457207	0.234529×10^{-2}	0.216143×10^{-1}	16
−0.2934327257	0.175721×10^{-5}	0.213237×10^{-3}	20
−0.2934327661	0.157964×10^{-6}	0.284395×10^{-6}	24
−0.2934327661	0.101174×10^{-8}	0.438053×10^{-8}	28
−0.2934327661	0.159340×10^{-11}	0.266255×10^{-10}	32
−0.2934327661	0.172756×10^{-13}	0.425269×10^{-14}	36

Note. N is the number of polynomials; $R_H = 114.2$, $R_V = 100$, $k = 0$, $m = 10$.

TABLE VI

The Real Part of the Ninth Eigenvalue $\sigma^{(9)}$ against the Number of Polynomials N; $R_H = 114.2$, $R_V = 100$, $k = 0$, $m = 10$

$\sigma_r^{(9)}$	N
$-0.1489144316 \times 10^4$	12
$-0.8984095537 \times 10^3$	16
$-0.8927935821 \times 10^3$	20
$-0.8927979712 \times 10^3$	24
$-0.8927979750 \times 10^3$	28
$-0.8927979750 \times 10^3$	32
$-0.8927979750 \times 10^3$	36

of the components equals one and the component of largest modulus is real. The convergence is evident from Tables IV and V, where also the τ coefficients are given. We remark that a very useful convergence indicator is simply the eigenvector yielded by the QZ algorithm, as is evident from Table IV. Table VI is included to show convergence of a higher eigenvalue.

It is clearly seen that $\sigma^{(1)}$ has converged to 10 digits with 24 polynomials, whereas $\sigma^{(9)}$ requires 28. The agreement between the two methods is perfect for large enough ODE tolerance.

The Eigenvalue Problem of Nonlinear Energy Stability Theory for Nield's Inclined Temperature Gradient Problem

We remark that the methods discussed here may be applied to solve the eigenvalue problem which arises with a nonlinear energy stability analysis of a porous convection problem. To give an example of this we briefly consider such an analysis for the problem under investigation in this subsection. Since the goal of this paper is not to discuss energy methods explicitly we stress that we do not here attempt to obtain the best nonlinear energy stability result; i.e., we do not optimize the energy problem.

The nondimensionalised fully nonlinear perturbation equations for (4.1) may be derived using Nield's [14] theory and are

$$\begin{aligned} \pi_{,i} &= -u_i + \delta_{i3}\theta, \\ u_{i,i} &= 0, \\ \theta_{,t} + u_i\theta_{,i} &= R_H u - U\frac{\partial\theta}{\partial x} - \frac{d\overline{T}}{dz}w + \Delta\theta, \end{aligned} \tag{4.6}$$

where the spatial domain is the three-dimensional layer $z \in (-\frac{1}{2}, \frac{1}{2})$, and this perturbation is subject to the boundary conditions

$$w = \theta = 0, \tag{4.7}$$

where the functions (u_i, θ, π) satisfy a plane tiling planform in the (x, y) plane. A typical "periodic cell" so formed is denoted by V.

We shall here deal only with an energy analysis which multiplies $(4.6)_1$ by u_i and $(4.6)_3$ by θ, although a sharper stability boundary may possibly be derived with a more general analysis, cf. [20]. In the ensuing analysis we need the relations

$$\begin{aligned} \langle u_i\theta_{,i}\theta\rangle &= 0, \\ \left\langle U\theta\frac{\partial\theta}{\partial x}\right\rangle &= 0, \end{aligned} \tag{4.8}$$

as may be shown by integrating by parts and use of the boundary conditions. Here $\langle\cdot\rangle$ denotes integration over a period cell V of the perturbation solution. Due to $(4.8)_2$ the base velocity $U(z)$ is lost in the resulting energy eigenvalue problem. It may well be that a more sophisticated energy analysis employing a weight, or even a weighted L^p functional, will retain U and yield a sharper result. We do not pursue this matter here; weighted and generalised energy methods are discussed in several contexts pertaining to convection in fluids and in porous media in [20].

Let now $\|\cdot\|$ denote the $L^2(V)$ norm. The energy identities we obtain from (4.6) are

$$\|\mathbf{u}\|^2 = \langle\theta w\rangle \tag{4.9}$$

and

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2 = R_H\langle u\theta\rangle - \langle(D\overline{T})w\theta\rangle - \|\nabla\theta\|^2. \tag{4.10}$$

Equation (4.10) is added to λ times (4.9) for a positive constant λ which may be judiciously selected to obtain the sharpest stability boundary. Let us denote by $E(t)$, $I(t)$, and $D(t)$,

$$E(t) = \tfrac{1}{2}\|\theta\|^2, \tag{4.11}$$

$$I(t) = R_H\langle u\theta\rangle - \langle(D\overline{T})w\theta\rangle + \lambda\,\langle\theta w\rangle, \tag{4.12}$$

$$D(t) = \|\nabla\theta\|^2 + \lambda\|\mathbf{u}\|^2. \tag{4.13}$$

The resulting energy equation is

$$\frac{dE}{dt} = I - D, \tag{4.14}$$

and if we define Λ by

$$\frac{1}{\Lambda} = \max_{\mathscr{H}} \frac{I}{D}, \tag{4.15}$$

where $\mathscr{H}$ is the space of admissible solutions, then from (4.14) we derive

$$\frac{dE}{dt} \le -D(1 - \Lambda^{-1}). \tag{4.16}$$

From Poincaré's inequality there exists a positive constant ξ such that

$$D \geq \xi E,$$

so that, provided

$$\Lambda < 1, \tag{4.17}$$

from (4.16) we may show

$$E(t) \le E(0) \exp\left[-\frac{\xi}{\Lambda}(1 - \Lambda)t\right]. \tag{4.18}$$

Inequality (4.18) guarantees strong decay of all disturbances and thus (4.17) is a criterion for unconditional (i.e., for all initial data) nonlinear energy stability.

To use (4.17) we must solve (4.15) and the Euler–Lagrange equations for this are

$$\begin{gathered}\Lambda[R_H\theta\delta_{i1} - (D\overline{T})\theta\delta_{i3} + \lambda\theta\delta_{i3}] - 2\lambda u_i = \pi_{,i},\\ u_{i,i} = 0, \\ \Lambda[R_H u - (D\overline{T})w + \lambda w] + 2\,\Delta\theta = 0,\end{gathered} \tag{4.19}$$

where π is here a Lagrange multiplier which arises due to the constraint $(4.19)_2$. The function π is eliminated from (4.19) and normal modes are introduced as in (4.2); Λ is chosen equal to one as this is the threshold and then we derive the system of equations:

$$\begin{gathered}(D^2 - a^2)W + \frac{1}{2}a^2[1 - \lambda^{-1}(D\overline{T})]\Theta + \frac{1}{2}ik\lambda^{-1}R_H D\Theta = 0,\\ (D^2 - a^2)\Theta + \frac{1}{2}(\lambda - D\overline{T})W + \frac{ik}{2a^2}R_H DW + \frac{R_H^2 m^2}{4\lambda a^2}\Theta = 0.\end{gathered} \tag{4.20}$$

Recall that

$$D\overline{T} = -R_V + R_H^2\left(\frac{1}{24} - \frac{1}{2}z^2\right),$$

(4.20) are defined on $z \in (-\frac{1}{2}, \frac{1}{2})$, and

$$W = \Theta = 0, \quad z = \pm\tfrac{1}{2}. \tag{4.21}$$

Unlike the eigenvalue problem in linear stability, σ is not present. Instead we fix R_H, k, m and solve for R_V. Of course, R_V must be real. This is so because Eqs. (4.20) arise from the variational problem (4.15), although it may be seen directly by assuming $R_V \in \mathbf{C}$. Then multiplying $(4.20)_1$ by W^* (complex conjugate), $(4.20)_2$ by Θ^*, and integrating over $(-\frac{1}{2}, \frac{1}{2})$. If we add the results and take the imaginary part we obtain

$$\tfrac{1}{2}a^2[\langle\Theta W^*\rangle + \langle W\Theta^*\rangle]R_V^i = 0, \tag{4.22}$$

where $\langle\cdot\rangle$ denotes integration over $(-\frac{1}{2}, \frac{1}{2})$ and $R_V = R_V^r + iR_V^i$. The coefficient of R_V^i in (4.22) is real and, so, unless W_r, Θ_r and W_i, Θ_i (the real and imaginary parts of W and Θ) are orthogonal then $R_V \in \mathbf{R}$. (If $R_H = 0$ we can show they are not orthogonal and evidently a continuation argument extends this to the general case.)

To completely determine the energy stability threshold we should now carry out the optimization

$$R_V^{\text{opt}} = \max_{\lambda} \min_{k,m} R_V(k, m; \lambda),$$

on the leading eigenvalue R_V of (4.20). For the linear problems we have conveniently carried out not dissimilar optimizations using the compound matrix method. In the tables below we present some results for the leading eigenvalue R_V for fixed k, m, λ obtained with the Chebyshev scheme solving the matrix problem

$$(A_r + iA_i)\mathbf{x} = R_V(B_r + iB_i)\mathbf{x},$$

where $\mathbf{x} = (W_0, ..., W_N, \Theta_0, ..., \Theta_N)$, and

$$A_r = \begin{pmatrix} D^2 - a^2 I & \dfrac{1}{2}a^2\left(1 - \dfrac{R_{\rm H}^2}{24\lambda}\right)I + \dfrac{R_H^2 a^2 z^2}{16\lambda} \\ \left(\dfrac{\lambda}{2} - \dfrac{R_H^2}{48}\right)I + \dfrac{R_H^2 z^2}{16} & D^2 - a^2 I + \dfrac{R_H^2 m^2}{4\lambda a^2}I \end{pmatrix}$$

$$A_i = \begin{pmatrix} 0 & \dfrac{kR_H}{2\lambda}D \\ \dfrac{kR_H}{2a^2}D & 0 \end{pmatrix}$$

$$B_r = \begin{pmatrix} 0 & -\dfrac{a^2}{2\lambda}I \\ -\dfrac{1}{2}I & 0 \end{pmatrix}$$

TABLE VII

R_V against N

R_V	N
44.24040165	10
44.24040292	12
44.24040294	14

Note. Here $R_H = 20$, $k = 0$, $m = 3.15$, $\lambda = 49.5$.

with $B_i = 0$ and where we have converted to the Chebyshev domain $(-1, 1)$.

With $N = 60$, $R_H = 114.2$, $k = 0$, $m = 18$, $\lambda = 100$ we find $R_V = 16.8324$. Of course, this is far from the linear value of $R_V = 100$, but no optimization has been performed.

Tables VII and VIII refer to the energy eigenvalue problem (4.20). Table VII demonstrates convergence in N. The value for R_V with $N = 14$ in Table VII was verified with $N = 20, 30, 40, 50, 60$. Nield [14] gives a value of $R_V = 49.56$ as the critical value according to linear theory and so Table VII shows that energy theory is likely to be very sharp for R_H not too large.

Remarks. While we have only presented one representative problem in this section we could easily have included many more, including new convection studies. For example, we could treat convection in snow, Powers *et al.* [19]; anisotropic porous convection, Straughan and Walker [22]; or several other novel problems of convection discussed in the texts of Nield and Bejan [15], and Straughan [20, 21].

5. CONCLUDING REMARKS

1. It is important to realize that the Chebyshev method extends to arbitrarily large systems which may be found when many constituents are present and even chemical reactions are taking place; we do, however, advocate arranging the systems of differential equations in such a way that no derivatives higher than second-order appear to avoid roundoff error due to growth of the matrix coefficients. Such problems are important as is indicated in the work and references of Pearlstein *et al.* [18], Straughan and Walker [23]. Also, we can extend the method to more than one space dimension; such problems are again of importance (see, e.g., Kim and Pearlstein [8], Zebib [26]). When very large systems are encountered as they are when many constituents are present and the systems are stiff, or higher dimensions are considered, e.g., [8, 26], then the matrices become large and are precisely the domain for application of software libraries, such as ScaLAPACK, Dongarra and Walker [3], on massively parallel computers. The present writers are exploiting this to solve other problems in hydrodynamic stability such as the Poiseuille flow problem with many constituents, a problem of importance in stellar atmospheres and in the stratosphere. Such problems are inevitably difficult and certainly lead to difficult eigenvalue problems. However, we have found the rapid convergence of Chebyshev polynomials to be extremely useful and coupled to the ScaLAPACK software, which is capable of solving very large linear algebra problems, should prove capable of handling complicated and large stability problems.

TABLE VIII

Variation of R_V with m and λ

R_V	m	λ
44.24963830	2.8	49.5
44.12615768	2.9	49.5
44.10418487	3.0	49.5
44.17389530	3.1	49.5
44.12894795	3.0	49.0
44.15097589	3.0	48.5

Note. $R_H = 20$, $k = 0$, $N = 30$.

2. The Chebyshev method is ideally suited to tackle a problem such as the one of Nield *et al.* [16]. These writers tackle a problem analogous to the one discussed in Section 4; however, a salt field is also present and this too is subject to a horizontal variation on the boundaries. The eigenvalue problem will inevitably again be very sensitive to parameter changes and will be essentially of sixth order. We advocate the method outlined here for obtaining accurate results for this problem and others like it.

3. While both methods advocated here are very accurate a few words of comparison are in order. The compound matrix method is particularly easy to change from problem to problem, usually requiring change in only one subroutine, that containing the differential equations for the y's. Also, different boundary conditions are easily incorporated with the compound matrix method. Since only one eigenvalue is tracked the compound matrix method may be quicker if optimization in other parameters is required. The Chebyshev technique, on the other hand, is especially easy to utilize to generate eigenfunctions. In addition, the Chebyshev approach is easy to extend to higher order systems. For problems where mode crossing is experienced; i.e., the eigenvalue which is dominant in one area of parameter space is replaced by another in moving to some other domain of parameter space, then a technique like the Chebyshev one, which yields all eigenvalues, is vital. Such mode crossing in actual physical problems is encountered in [2, 18, 23]. A final point is that the Chebyshev tau method is particularly suitable for eigenvalue problems, where the differential equation contains a singular term like $(m/r)(d/dr)$. Such problems occur naturally in porous flows in a circular pipe, or convection in a porous sphere or shell. The relationship

$$T_{m-1}(z) + T_{m+1}(z) = 2zT_m(z),$$

e.g., Orszag [17], is the key to why the tau method is eminently suitable for application to such singular eigenvalue problems. Although the application is not to flow in a porous material, further details may be found in, e.g. [2], where the Poiseuille problem of flow of a viscous fluid in a pipe is examined by the tau method by writing the equations as a system of second-order ones.

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